AN OPEN MAPPING THEOREM

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Abstract. It is proved that any surjective morphism $f : \mathbb{Z}^\kappa \to K$ onto a locally compact group K is open for every cardinal κ. This answers a question posed by Karl Heinrich Hofmann and the second author.

1. Introduction

In this paper we assume that all topological groups are Hausdorff and abelian. In the literature it is common to ask whether a surjective continuous homomorphism $f : G \to K$ of a topological group G onto a topological group K is an open mapping. Positive results in this direction are known as "Open Mapping Theorems" in the literature in Functional Analysis and Topological Algebra (see, for example, [2, 2.25] for Banach spaces, [8] for Polish groups and [5, 9.60] and [4] for pro-Lie groups). Most results of this type impose a countability condition on G. Indeed, if K is any countable non-discrete group or an infinite compact one and $G := K_d$ is the group K endowed with the discrete topology, then the identity map $i : G \to K$ is not open. Noting that for every uncountable cardinal κ the totally disconneced abelian group $G = \mathbb{Z}^\kappa$ is neither a Polish group nor a locally compact group, K.H. Hofmann and the second author posed the following question, see Question 5 in [7]: *Is a surjective morphism $f : \mathbb{Z}^\kappa \to K$ onto a compact group open for every cardinal κ?* We answer this question in the affirmative.

We will use the following notation and terminology. For a topological group K, we denote by K_0 the connected component of the identity. A topological group K is called *almost connected* [5] if the quotient group K/K_0 is compact. A topological group G is called a *pro-Lie group* [5] if it is a closed subgroup of a product of finite-dimensional Lie groups. So the group $\mathbb{Z}^\kappa$ is a non-almost connected pro-Lie group. Every compact group is an almost connected pro-Lie group.

We denote by $\mathcal{CDA}$ the class of all totally disconnected abelian groups G such that for every open subgroup H of G the quotient group G/H is countable. Note that the class $\mathcal{CDA}$ is closed under taking subgroups, Hausdorff quotient groups and arbitrary products (with the Tychonoff topology).

2. Results

The first lemma is an immediate consequence of Proposition 5.43 of [5].

Lemma 1. *Every non-totally disconnected abelian pro-Lie group K has the circle group as a quotient group.*

Lemma 2. *Let $G \in \mathcal{CDA}$. If there is a surjective morphism $f : G \to K$ onto a pro-Lie group K, then K also belongs to the class $\mathcal{CDA}$.*

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Proof. Suppose K is not totally disconnected. Then, by Lemma 1, there is a continuous homomorphism $\bar{f}$ from G onto $\mathbb{T}$. Let U be any small neighborhood of the identity in the circle group $\mathbb{T}$. Note that U contains no non-trivial subgroups of $\mathbb{T}$. As $\bar{f}^{-1}(U)$ is an open neighborhood of zero of G, $\bar{f}^{-1}(U)$ contains an open subgroup H of G such that G/H is countable. So $f(H) = \{0\}$ and hence $f(G) = \mathbb{T}$ is also countable, a contradiction to our supposition. Hence G is totally disconnected.

It remains to show that for every open subgroup H of K the quotient group K/H is countable. This follows from the fact that $G/f^{-1}(H)$ is countable and f is surjective. $\square$

Lemma 3. *Let K be an almost connected abelian pro-Lie group which is either totally disconnected or a torsion group. Then K is compact.*

Proof. If K is totally disconnected, then $K = K/K_0$ is compact by definition.

Assume that K is torsion. By Theorem 5.20 of [5], a torsion abelian connected pro-Lie group is compact. So the connected component K_0 of K is compact. But the almost connected group K by definition is then an extension of a compact group by a compact group. As compactness is a three space property for topological groups, K is compact. $\square$

For every $m > 1$ and cardinal number κ, the group $\mathbb{Z}^\kappa/m\mathbb{Z}^\kappa = \mathbb{Z}(m)^\kappa$ is compact. Being motivated by this fact, we denote by $\mathcal{CDA}_k$ the class of all groups $G \in \mathcal{CDA}$ for which G/G_m is compact for every natural number $m > 1$, where $G_m := \mathrm{cl}_G(mG)$, the closure in G of mG. So $\mathbb{Z}^\kappa \in \mathcal{CDA}_k$. Note that the group G/G_m has order $\leq m$, for every $m > 1$. We note also that the class $\mathcal{CDA}_k$ is closed under taking Hausdorff quotient groups and arbitrary products.

Lemma 4. *Let $G \in \mathcal{CDA}_k$. If $f : G \to K$ is a surjective continuous homomorphism onto an almost connected torsion pro-Lie group K, then f is an open mapping.*

Proof. By Lemma 3 we can suppose that K is a compact abelian group.

Since K is torsion, there is an $m \in \mathbb{N}$ such that $mK = 0$ by Theorem 25.9 of [3]. Then the closed subgroup G_m of G is contained in the kernel, $\ker(f)$, of f. So f induces an injective continuous homomorphism $\tilde{f}$ from $G/\ker(f) = \big(G/G_m\big)/\big(\ker(f)/G_m\big)$ onto K. As G/G_m is a compact group, we obtain that $G/\ker(f)$ is also compact. Hence $\tilde{f}$ is a topological group isomorphism of the compact group $G/\ker(f)$ onto K. Since the projection $\pi : G \to G/\ker(f)$ is an open mapping, we obtain that $f = \tilde{f} \circ \pi$ is also an open mapping, as required. $\square$

We now recall two algebraic notions. An abelian group G is called *reduced* if G does not have non-trivial divisible subgroups. Clearly, *the group of all integers, $\mathbb{Z}$, is reduced.* An abelian group G is called *algebraically compact* if G is a direct summand of an abelian group which admits a compact group topology (see the Corollary in [1]).

To prove Theorem 6 we need the following lemma which is an immediate corollary of Theorem 6.4 of [9].

Lemma 5. *The group $\mathbb{Z}$ is not algebraically compact.*

Now we prove our main result.

Theorem 6. *Let K be a pro-Lie group which has an open almost connected subgroup H. For every cardinal κ, any surjective continuous homorphism $f : \mathbb{Z}^\kappa \to K$ is an open mapping.*

Proof. Without loss of generality we shall assume that the group H is infinite and hence the cardinal κ is also infinite. We split the proof into two steps.

Step 1. Assume that K is an almost connected pro-Lie group. By Lemmas 2 and 3, we can assume also that K is compact. It is enough to prove that the image $S := f(U)$ of an open subgroup $U = \{0_i\} \times \mathbb{Z}^{\kappa\setminus\{i\}}$ of $\mathbb{Z}^\kappa$ is open in K, for every $i \in \kappa$.

Set $e := f(1_i) \in K$ and let $\langle e\rangle$ be the cyclic subgroup of K generated by e. Note that, by hypothesis, $K = \langle e\rangle + S$. We have to show that S is open.

We claim that there is an $m \in \mathbb{N}$ such that $me \in S$. Suppose this is not the case, then we obtain that $\langle e\rangle \cap S = \{0\}$, and hence the subgroup $\langle e\rangle \cong \mathbb{Z}$ is a direct (algebraic) summand of the compact group K. So $\mathbb{Z}$ is an algebraically compact group which is false since it contradicts Lemma 5.

So let $m \in \mathbb{N}$ be such that $me \in S$. Then $mK \subset S$. Let $\pi : K \to K/mK$ be the quotient map. Since K/mK is torsion, Lemma 4 implies that the map $\bar{f} := \pi \circ f$ is open. So $\bar{f}(U)$ is open in K/mK. Hence the subgroup $f(U) = S = \pi^{-1}\big(\bar{f}(U)\big)$ is open in K. Thus f is an open mapping.

Step 2. Assume that K contains an open almost connected subgroup H. Since the subgroup $X := f^{-1}(H)$ of $\mathbb{Z}^\kappa$ is open, we can find a finite subset $F = \{i_1, \dots, i_n\}$ of κ such that X contains the open subgroup $Y := \mathbb{Z}^{\kappa\setminus F}$. Since X/Y is a subgroup of $\mathbb{Z}^n = \mathbb{Z}^\kappa/Y$, there is a $k \in \mathbb{N}$ such that $X/Y = \mathbb{Z}^k$ by [3, A 26].

As the projection π_Y of X onto Y is continuous and $\pi_Y(y) = y$, for every $y \in Y$, we obtain that $X = X/Y \times Y$, see [3, 6.22]. So X is topologically isomorphic to $\mathbb{Z}^k \times Y$. Hence the restriction map $p := f|_X$ from X onto H is open by Step 1. As H is open, we obtain that f is also an open mapping, as required. □

The Principal Structure Theorem for Locally Compact Abelian Groups (Theorem 25 of [10]) says that every locally compact abelian group K has an open subgroup H which is topologically isomorphic to $\mathbb{R}^n \times C$, where C is a compact abelian group and n is a non-negative integer. So H is an almost connected pro-Lie grup. So as an immediate consequence of Theorem 6 we obtain Corollary 7, which provides a positive answer to Question 5 of [7].

Corollary 7. *Let K be a locally compact abelian group. For every cardinal κ, any surjective continuous homorphism $f : \mathbb{Z}^\kappa \to K$ is an open mapping. In particular, this is the case if K is compact.*

Indeed, since a pro-Lie group K with the property that K/K_0 is locally compact has an open subgroup which is an almost connected pro-Lie group by Corollary 8.12 of [6], we obtain the stronger result:

Corollary 8. *Let K be an abelian pro-Lie group K with the property that K/K_0 is locally compact. For every cardinal κ, any surjective continuous homorphism $f : \mathbb{Z}^\kappa \to K$ is an open mapping.*

We conclude with an open question.

Question 9. *Is every surjective continuous homomorphism from $\mathbb{Z}^\kappa$ onto a pro-Lie group K open?*

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